Chapter 0

Power Series Continued

In Section 1 Cauchy-Hadamard Theorem on the radius of convergence is reviewed and Abel's Limit Theorem is proved. Infinite product is taken up in Section 2 and then applied to the proof of Newton's Binomial Theorem in Section 3. The chapter ends with Euler's formulas on the sum of negative powers in Section 4.

0.1 Cauchy-Hadamard Theorem

Recall a series of functions is of the form $\sum_{j=j_0}^{\infty} f_j(x)$ where each f_j is a function defined on a common subset E of $\mathbb{R}$. In practice there are two kinds of series of functions which are important: Power series and trigonometric series. Power series was discussed in brief at the end of MATH2060. Here we present further results.

By a **power series** we mean a series of the form $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ where $a_j \in \mathbb{R}$ and x_0 is a fixed point in $\mathbb{R}$. For instance,

$$\sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

and

$$\sum_{j=3}^{\infty} j(x-2)^j = 3(x-2)^3 + 4(x-2)^4 + 5(x-2)^5 + \cdots ;$$

are power series. In the second example, it is understood that all $a_j = 0$ for odd j, and in the last example, $a_0 = a_1 = a_2 = 0$. One should keep in mind that inserting zeros between two summands of a series affects neither the convergence nor the final sum of the series.

Given a series of functions, we would like to determine its pointwise convergence and uniform convergence. There are numerous tests for pointwise convergence, for instance, the ratio, root, Raabe's, integral tests for absolute convergence and Alternating, Dirichlet's and Abel's tests for conditional convergence. For uniform convergence, Weierstrass M-Test and Cauchy criterion are the common tools. However, for power series we have a very general and yet precise result. To formulate it one needs to introduce the notion of the radius of convergence of a power series.

Let

$$\rho := \overline{\lim_{n\to\infty}} |a_n|^{1/n} \in [0,\infty] \ .$$

Define the **radius of convergence** of $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ to be

$$R = \begin{cases} 0, & \text{if } \rho = \infty, \\ 1/\rho, & \text{if } \rho \in (0,\infty), \\ \infty, & \text{if } \rho = 0. \end{cases}$$

The following theorem is the main result for power series.

Theorem 0.1. ***(Cauchy-Hadamard Theorem)***

(a) When $R \in (0,\infty)$, the power series $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ converges absolutely at every $x \in (x_0 - R, x_0 + R)$ and diverges at every x satisfying $|\,x - x_0\,| > R$. Moreover, the convergence is uniform on every subinterval $[a,b] \subset (x_0 - R, x_0 + R)$.

(b) When $R = \infty$, the power series converges absolutely at every $x \in \mathbb{R}$ and converges uniformly on any finite interval.

(c) When $R = 0$, the power series diverges at every $x \in \mathbb{R} \setminus \{x_0\}$.

Proof. (a) We show that for any $) < r < R$, the series is absolutely and uniformly convergent on $[x_0 - r, x_0 + r]$. To this end we fix a small $\delta > 0$ such that $(\rho+\delta)r < 1$. This is possible because $\rho r = r/R < 1$. Then, as $\overline{\lim} \sqrt[n]{|\,a_n\,|} = \rho$, there exists n_0 such that $\sqrt[n]{|\,a_n\,|} \le \rho + \delta$, $\forall\, n \ge n_0$. For $x \in [x_0 - r, x_0 + r]$, we have

$$\begin{aligned} \sqrt[n]{|\,a_n(x-x_0)^n\,|} &= \sqrt[n]{|\,a_n\,|}\,|\,x - x_0\,| \\ &\le \sqrt[n]{|\,a_n\,|}\,r \\ &\le (\rho+\delta)r < 1, \quad \forall\, n \ge n_0. \end{aligned}$$

Taking $\alpha = (\rho+\delta)r$, we have $|\, a_n(x-x_0)^n \,| \le \alpha^n$ and $\sum_{n=0}^{\infty} \alpha^n < +\infty$. By Weierstrass M-Test we conclude that $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ converges absolutely and uniformly on $[x_0 - r, x_0 + r]$.

When x_1 satisfies $|x_1 - x_0| > R$, assume it is $x_1 - x_0 > R$, say. Fix an $\varepsilon_0 > 0$ such that $(\rho - \varepsilon_0)(x_1 - x_0) > 1$. This is possible because $\rho(x_1 - x_0) > \rho R = 1$. There exists n_1 and a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $|a_{n_j}|^{1/n_j} \ge \rho - \varepsilon_0$, $\forall\ n_j \ge n_1$. Then

$$\sqrt[n_j]{|\, a_{n_j}(x_1 - x_0)^{n_j} \,|} = |\, a_{n_j} \,|^{1/n_j}(x_1 - x_0) \ge (\rho - \varepsilon_0)(x_1 - x_0) > 1.$$

It shows that $\{\, a_n(x_1 - x_0)^n \,\}$ does not tend to zero, so the power series diverges.

The proofs of (b) and (c) can be obtained from modifying the above proof. We omit them.

□

In passing, we point out another way to evaluate the radius of convergence is by the formula

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

provided the limit exists.

Recall that termwise differentiation and integration of a given power series yield power series:

$$\sum_{j=0}^{\infty} (j+1)a_{j+1}x^j \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{a_{j-1}}{j} x^j.$$

It is routine to check that these two series have the same radius of convergence as the original one. (Indeed, suppose that $\rho = \lim_{j\to\infty} |a_j|^{1/j}$ exists and belongs to $(0,\infty)$. For $\varepsilon > 0$, there exists some n_0 such that $\rho - \varepsilon/2 \le |a_j|^{1/j} \le \rho - \varepsilon/2$, for all $n \ge n_0$. It follows that $(\rho - \varepsilon/2)^{1+1/j} \le |a_{j+1}|^{1/j} \le (\rho + \varepsilon/2)^{1+1/j}$. As $\lim_{j\to\infty}(\rho + \varepsilon/2)^{1/j} = 1$, we can find some $n_1 \ge n_0$ such that $(\rho + \varepsilon/2)^{1+1/j} \le \rho + \varepsilon$ and $\rho - \varepsilon \le (\rho - \varepsilon/2)^{1+1/j}$ for all $n \ge n_1$. Therefore, for these n, $|a_{j+1}^{1/j} - \rho| < \varepsilon$. We have shown that $\lim_{j\to\infty} |a_{j+1}|^{1/j} = \rho$. The j-th term in the series obtained from differentiation is given by $(j+1)a_{j+1}$. We have $\lim_{j\to\infty} |(j+1)a_{j+1}|^{1/j} = \lim_{j\to\infty} |j+1|^{1/j} \lim_{j\to\infty} |a_{j+1}|^{1/j} = \lim_{j\to\infty} |a_{j+1}|^{1/j} = \rho$, so this derived series has the same radius of convergence as the original one. The other cases can be treated similarly.) Since the partial sums of a power series are polynomials, in particular they are continuous on $[x_0 - r, x_0 + r]$ for $r < R$. By uniform convergence the power series is a continuous function on $[x_0 - r, x_0 + r]$ and hence on $(x_0 - R, x_0 + R)$. In fact, we have

Theorem 0.2. *Every power series is a smooth function on $(x_0 - R, x_0 + R)$. Moreover, termwise differentiations and integrations commute with the summation.*

Proof. Letting

$$f(x) \equiv \sum_{j=0}^{\infty} a_j (x-x_0)^j, \quad x \in (x_0 - R, x_0 + R) \ ,$$

and

$$g(x) \equiv \sum_{j=0}^{\infty} (j+1) a_{j+1} (x-x_0)^j, \quad x \in (x_0 - R, x_0 + R) \ ,$$

which is obtained from termwise differentiating f. Since both power series uniformly converge to f and g respectively on $[x_0 - r, x_0 + r], r \in (0, R)$, by the exchanged theorem in MATH2060 we conclude that f is differentiable and $f' = g$. Repeating this argument, we arrive at the conclusion that f is smooth in $(x_0 - R, x_0 + R)$ and termwise differentiations are commutative with the summation. A similar argument established the case for termwise integration. $\square$

Cauchy-Hadamard theorem says nothing on the convergence of a power series at its "boundary points". Let us consider an example.

Example 0.1. We start with the "mother" geometric series

$$\sum_{j=0}^{\infty} (-1)^j x^j = 1 - x + x^2 - x^3 + x^4 - \cdots$$

it is clear that its radius of convergence is equal to 1. Integrating both sides from 0 to $x \in (-1, 1)$, we get the "grandmother"

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} x^j}{j} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ .$$

Integrating once more yields the "great grandmother"

$$\sum_{j=2}^{\infty} \frac{(-1)^{j-2} x^j}{(j-1)j} = \frac{x^2}{2} - \frac{x^3}{2 \times 3} + \frac{x^4}{3 \times 4} - \frac{x^5}{4 \times 5} + \cdots \ .$$

On the other hand, by differentiating the mother we get the "child"

$$\sum_{j=0}^{\infty} (-1)^{j+1} (j+1) x^j = -1 + 2x - 3x^2 + 4x^3 - \cdots \ .$$

According to Cauchy-Hadamard theorem these series are all convergent in $(-1, 1)$ and divergent in $(-\infty, -1) \cup (1, \infty)$. At the boundary points 1 and -1 they could be convergent or divergent. Indeed, both the "mother" and the "child" diverge at the boundary points, the "grandmother" converges at 1 but diverges at -1, and the "great grandmother" converges at both ends. In general, when one family member is convergent at a boundary point, its ancestor is also convergent at the same point. (Why?) But the converse is not necessarily true.

From this example you can see that the convergence of a power series at the boundary points is a delicate matter and must be examined case by case. We will present a general result, namely, Abel's limit theorem. It can be deduced from a variation of Abel's criterion. Last year we learned this criterion for the conditional convergence of series of numbers. Now we extend it to uniform convergence of series of functions.

Theorem 0.3. *Consider* $\sum_n f_n g_n$ *where* $\sum_n g_n$ *is uniformly convergent on* $E \subset \mathbb{R}$ *and* $f_n(x)$ *is monotone for each* $x \in E$ *and uniformly bounded. Then* $\sum_n f_n g_n$ *is uniformly convergent on* E.

Proof. Without loss of generality assume that f_n is decreasing. Let $M > 0$ satisfy $|f_n(x)| \leq M$ for all n and $x \in E$. For $\varepsilon > 0$, there is some n_0 such that $\|\sigma_m - \sigma_n\| < \varepsilon$ for all $m, n \geq n_0$, where

$$\sigma_n = \sum_{j=n_0}^{n} g_j \ ,$$

and $\|\cdot\|$ refers to the sup-norm over E. For $m, n \geq n_0$,

$$\begin{aligned}
\left|\sum_{j=1}^{m} f_j g_j - \sum_{j=1}^{n} f_j g_j\right| &= \left|\sum_{j=n+1}^{m} f_j g_j\right| \\
&= \left|\sum_{j=n+1}^{m} f_j(\sigma_j - \sigma_{j-1})\right| \\
&= \left|\sum_{j=n+1}^{m} f_j \sigma_j - \sum_{j=n+1}^{m-1} f_{j+1}\sigma_j\right| \\
&= \left|\sum_{j=n+1}^{m-1} (f_j - f_{j+1})\sigma_j + f_m\sigma_m - f_{n+1}\sigma_n\right| \\
&\leq \varepsilon(|f_m| + |f_{m-1}|) \\
&\leq \frac{2\varepsilon}{M} \ .
\end{aligned}$$

This is valid for all x in E. Therefore, $\sum_{j=1}^{n} f_j g_j$ forms a Cauchy sequence in sup-norm. By Cauchy's Criterion, $\sum_{j=1}^{n} f_j g_j$ converges uniformly on E.

□

Theorem 0.4. ***(Abel's Limit Theorem)*** *Let the radius of convergence of* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ *be* $R \in (0, \infty)$. *If* $\sum_{n=0}^{\infty} a_n R^n$ *is convergent, then* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ *converges uniformly on* $(a, R]$ *for each* $a \in (-R, R)$. *If If* $\sum_{n=0}^{\infty} a_n(-R)^n$ *is convergent, then* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ *converges uniformly on* $[-R, b)$ *for each* $b \in (-R, R)$.

Proof. The theorem follows from Theorem 1.3 by taking $g_n(x) = a_n R^n$ and $f_n(x) = R^{-n}(x - x_0)^n$ in the first case.

□

Applying this theorem to the series in Example 1.1.

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}x^j}{j} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots ,$$

we conclude that it is uniformly convergent on $[a, 1]$ for any $a \in (-1, 1)$. On the other hand, it is not uniformly convergent on $[-1, 1]$ since it diverges at $x = -1$.

We will show some interesting identities can be derived from convergence at the boundary points. Indeed, let us consider the elementary identity

$$\frac{1}{1+x} = \sum_{j=0}^{\infty} (-1)^j x^j, \quad x \in (-1, 1).$$

First of all, by integrating both sides of this identity from 0 to $x \in (0, 1)$, we get

$$\log(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}x^j}{j}, \quad x \in (-1, 1).$$

At $x = 1$, the series $\sum_{j=1}^{\infty} \dfrac{(-1)^{j+1}}{j}$ is convergent, it follows from the continuity of the logarithmic function at 1 and Abel's Limit Theorem that

$$\lim_{x\to 1^-} \log(1+x) = \lim_{x\to 1^-} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}x^j}{j} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}.$$

In other words, we have

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Furthermore, replace x in the identity by x^2 ($x^2 < 1$ whenever $|x| < 1$). Then

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} (-1)^j x^{2j}, \quad x \in (-1, 1) .$$

Integrating once yields

$$\text{Arctan } x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{2j+1} .$$

Letting $x \to 1^-$ and by Abel's Limit Theorem again, we obtain the following formula which was discovered by Leibniz:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

0.2 Infinite Product

We make a degression to study infinite products. They will be used in subsequent sections.

Let $\{a_n\}$ be a sequence of positive numbers. We form the infinite product $\prod_{n=1}^{\infty} a_n$. It is called **convergent** if its n-th **partial product**, $p_n = \prod_{j=1}^{n} a_n$, is a convergent sequence with *positive limit*. We denote this limit by $\prod_{n=1}^{\infty} a_n$. You may wonder why zero limit is excluded in the definition. Indeed, the sole purpose is to make the following proposition holds.

Proposition 0.5. *Let $a_n > 0$ and p_n be the n-th partial product of the infinite product $\prod_{n=1}^{\infty} a_n$. The infinite product is convergent if and only if the infinite series $\sum_{n=1}^{\infty} \log a_n$ is convergent. When this holds,*

$$\log \prod_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \log a_n \ .$$

Proof. Let $s_n = \sum_{j=1}^{\infty} \log a_j$ be the n-th partial sum of $\sum_{n=1}^{\infty} \log a_n$. Then $s = \sum_{n=1}^{\infty} \log a_n$ is convergent if and only if $s = \lim_{n\to\infty} s_n$ exists. By the continuity of the exponential function, it implies that

$$\lim_{n\to\infty} p_n = e^{\lim_{n\to\infty} s_n} = e^s > 0 \ ,$$

so $\prod_{n=1}^{\infty} a_n$ is convergent. Conversely, using the continuity of the logarithmic function on $(0, \infty)$, we have

$$\sum_{n=1}^{\infty} \log a_n = \lim_{n\to\infty} s_n = \log(\lim_{n\to\infty} p_n) = \log \prod_{n=1}^{\infty} a_n \ .$$

□

Corollary 0.6. *For a convergent infinite product $\prod_{n=1}^{\infty} a_n$, $\lim_{n\to\infty} a_n = 1$.*

In the following we let $b_n = a_n - 1$ so that $b_n > 0$ if and only if $a_n > 1$.

Theorem 0.7. *Consider $\prod_{n=1}^{\infty} a_n$.*

(a) Let $a_n > 0$ after some terms. Suppose that $\sum_{n=1}^{\infty} b_n^2$ is convergent. The infinite product is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is convergent.

(b) Let $a_n > 1$ after some terms. The infinite product is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is convergent.

Proof. (a) Here we use

$$\lim_{t\to 0} \frac{\log(1+t)}{t} = 1 \ .$$

For $\varepsilon = 1/2$, there is a corresponding δ such that

$$\left| \frac{\log(1+t)}{t} - 1 \right| < \frac{1}{2}, \quad \forall t,\ |t| < \delta \ ,$$

that is,

$$\frac{t}{2} < \log(1+t) < \frac{3t}{2} \ , \quad \forall t,\ 0 < t < \delta.$$

It follows form $n_0 > 1/\delta$,

$$\frac{1}{2}\sum_{j=n_0}^{n} b_n \le \sum_{j=n_0}^{n} \log(1+b_n) \le \frac{3}{2}\sum_{j=n_0}^{n} b_n \ ,$$

from which the desired result follows.

(b) Observe that

$$\lim_{t\to 0} \frac{t - \log(1+t)}{t^2} = \frac{1}{2} \ .$$

For $\varepsilon = 1/4$, there is a corresponding δ such that

$$\left| \frac{t - \log(1+t)}{t^2} - \frac{1}{2} \right| < \frac{1}{4}, \quad \forall t, |t| < \delta \ ,$$

that is,

$$\frac{t^2}{4} < t - \log(1+t) \le \frac{3t^2}{4} \ , \quad \forall t, |t| < \delta \ .$$

It follows for $n_0 > 1/\delta$,

$$\frac{1}{4}\sum_{j=n_0}^{n} b_n^2 \le \sum_{j=n_0}^{n} b_n - \sum_{j=n_0}^{n} \log(1+b_n) \le \frac{3}{4}\sum_{j=n_0}^{n} b_n^2 \ ,$$

from which the desired result follows. □

Example 0.2. Study the convergence of the following three infinite products:

(a) $a_n = \dfrac{1}{n}$,

(b) $b_n = 1 + \dfrac{b}{n}, \quad b > 0.$

(c) $c_n = 1 - e^{-n}$.

(a) We have $p_n = 1/n!$ which clearly tends to 0. In other words, we have $\prod_{n=1} 1/n = 0$. According to the definition of convergence for an infinite product, although the partial products converge, the infinite product is divergent because the limit is 0.

(b) By Theorem 1.6(a) and $\sum_n 1/n = \infty$ we conclude that $\prod_n (1 + b/n)$ is divergent.

(c) It is clear that both $\sum_n e^{-n}$ and e^{-2n} are convergent. Therefore, $\prod_n (1 - e^{-n})$ is also convergent by Theorem 1.6(b).

0.3 Newton's Binomial Theorem

Binomial Theorem states that

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j \ ,$$

for all $n \geq 0$. Newton found the analogous formula when n is replaced by an arbitrary real number. In this section we prove Newton's general formula.

For any real number α, consider the power series

$$\sum_{j=0}^{\infty} c_j x^j,$$

where

$$c_j = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-j+1)}{j!}, \quad j \in \mathbb{N},$$

and $c_0 = 1$. We call this power series a **binomial series** and c_n the **n-th binomial coefficient** of the binomial series. When $\alpha \in \{0, 1, 2, 3, \cdots\}$, this power series becomes a polynomial since all c_j's vanish after a particular j . In the following we consider $\alpha \in \mathbb{R} \setminus \{0, 1, 2, \cdots\}$ so that it has infinitely many non-zero binomial coefficients.

Theorem 0.8. *For $\alpha \in \mathbb{R} \setminus \{0, 1, 2, \cdots\}$, the radius of convergence of any binomial series is 1. Moreover, it*

(i) converges absolutely at $x = \pm 1$ when $\alpha > 0$,

(ii) diverges at $x = \pm 1$ when $\alpha \leq -1$,

(iii) converges conditionally at $x = 1$ and diverges at $x = -1$ when $\alpha \in (-1, 0)$.

Proof. We have
$$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{\alpha-n}\right| = 1.$$
It follows that the radius of convergence is equal to 1.

When $\alpha>0$ at $x=\pm1$, we have, for $n>\alpha+1$
$$\left|\frac{c_{n+1}x^{n+1}}{c_nx^n}\right| = \frac{n-\alpha}{n+1} = 1-\frac{\alpha+1}{n+1}.$$

By the limit form of Raabe's Test, the binomial series converges absolutely at $x=\pm1$.

Next, when $\alpha\le -1$ and $x=\pm1$,
$$\begin{aligned}|c_nx^n| = |c_n| &= \frac{|\alpha|}{1}\frac{(|\alpha|+1)}{2}\frac{(|\alpha|+2)}{3}\cdots\frac{(|\alpha|+n-1)}{n}\\ &\ge |\alpha|>1,\end{aligned}$$

so the binomial series is divergent.

Finally, consider $\alpha\in(-1,0)$ and $x=-1$. In this case every term of this series is positive, and we have
$$\begin{aligned}c_nx^n &= |c_n|\\ &= |\alpha|\frac{(|\alpha|+1)}{1}\frac{(|\alpha|+2)}{2}\cdots\frac{(|\alpha|+n-1)}{n-1}\frac{1}{n}\\ &\ge |\alpha|\frac{1}{n}.\end{aligned}$$
As $\sum 1/n=\infty$, the series is also divergent by the comparison test.

When $x=1$, the series is alternating. Observe that
$$\begin{aligned}|c_n| &= \left(1-\frac{1+\alpha}{1}\right)\left(1-\frac{1+\alpha}{2}\right)\cdots\left(1-\frac{1+\alpha}{n}\right)\\ &= \prod_{k=1}^{n}\left(1-\frac{1+\alpha}{k}\right).\end{aligned}$$
By Theorem 1.6(a) and the fact that $\sum_k 1/k=\infty$, we conclude that the series is divergent.

$\square$

Now we come to the main result of this section. Recall when $\alpha\in\{0,1,2,3,\cdots\}$, the binomial theorem
$$(1+x)^\alpha = \sum_{j=0}^{\alpha} c_jx^j,$$

where the binomial coefficients c_j's is defined in the previous section, has been known for a long long time. Notice that for any natural number $\alpha = n$, $c_j = \binom{n}{j}$, where $j \in \{0, 1, \ldots, n\}$. It was the insight of Newton who found the extension for other values of α.

Theorem 0.9. ***(Newton's Binomial Theorem)*** *For* $\alpha \in \mathbb{R} \setminus \{0, 1, 2, 3, \cdots\}$ *, we have*

$$(1+x)^\alpha = \sum_{j=0}^{\infty} c_j x^j, \quad \forall x \in (-1, 1), \tag{0.1}$$

where

$$c_j = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-j+1)}{j!}, \quad j \in \mathbb{N} \ .$$

The convergence is uniform on any $[a, b]$ *in* $(-1, 1)$*. In fact,*

(a) when $\alpha > 0$*, it is uniform on* $[-1, 1]$*,*

(b) when $\alpha \in (-1, 0)$*, it is uniform on* $[-a, 1]$*,* $a \in (0, 1)$*, and not uniform on* $[-1, 1]$*, and*

(c) when $\alpha \leq -1$*, it is not uniform up to either* 1 *or* -1*.*

The n-th Taylor's polynomial of the function $(1+x)^\alpha$ at the origin is precisely the n-th partial sum of the binomial series for α. Thus Taylor's Expansion Theorem provides a link between this function and the corresponding binomial series. In the following proof we see how the Taylor's Theorem with Integral Remainder works better than the Taylor's Theorem with Lagrange Remainder.

Proof. We consider positive values of x first. By Taylor's Expansion Theorem with Integral Remainder in MATH2060, for any $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha - \sum_{k=0}^{n} c_k x^k = (n+1)c_{n+1} \int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt.$$

If $x \in [0, b],\ b < 1$, we have, for $n + 1 > \alpha$,

$$\begin{aligned} &(n+1)|c_{n+1}| \int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt \\ &\qquad \leq (n+1)|c_{n+1}| \int_0^x (x-t)^n dt \leq |c_{n+1}| b^{n+1}. \end{aligned}$$

By Theorem 0.8, $\sum c_j b^j$ is convergent, so $|c_{n+1}|b^{n+1}$ tends to 0 as $n \to \infty$. Thus, for every $\varepsilon > 0$, there exists n_0 such that for any $\alpha \in \mathbb{R}$,

$$|(1+x)^\alpha - \sum_{j=0}^{n} c_j x^j| \leq |c_{n+1}| b^{n+1} < \varepsilon,\ \forall n \geq n_0,\ \forall x \in [0, b].$$

We have shown that for any real α, (0.1) holds and the convergence is uniform on $[0,b]$ for any $b<1$.

Next, consider negative values of x. When $x \in [a,0],\ a>-1$, we use the Mean-Value Theorem to get

$$\begin{aligned}\Big|c_{n+1}\int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt\Big| &= (n+1)|c_{n+1}|\int_x^0 (1+t)^{\alpha-n-1}(|x|+t)^n dt \\ &= (n+1)|c_{n+1}|(1+\xi x)^{\alpha-n-1}(|x|-\xi|x|)^n\int_x^0 dt \\ &= (n+1)|c_{n+1}|(1+\xi x)^{\alpha-1}\Big(\frac{1-\xi}{1+\xi x}\Big)^n|x|^{n+1},\end{aligned}$$

for some $\xi \in (0,1)$. As $x \in (-1,0),\ 1-\xi \le 1+\xi x$, so

$$\Big|c_{n+1}\int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt\Big| \le M(n+1)|c_{n+1}||x|^{n+1},$$

where $M = \sup\Big\{(1+\xi x)^{\alpha-1}\ :\ \xi \in (0,1),\ x \in [a,0]\Big\}$. By Theorem 0.8 the radius of convergence of $\sum c_j x^j$ is equal to one. It implies that the radius of convergence of the series $\sum(j+1)c_j x^j$ is also equal to one. Thus $\sum(j+1)c_j a^j$ converges absolutely and consequently $(n+1)c_{n+1}|a^{n+1}| \to 0$ as $n\to\infty$. As before we conclude that $\sum c_j x^j$ converges uniformly to $(1+x)^\alpha$ on $[a,0]$ for any real α.

We have shown that (1.1) holds where the convergence is uniform on any $[a,b], a,b \in (-1,1)$. Now uniform convergence assertions in (a), (b) and (c) follow from Theorem 1.8.

$\square$

0.4 Euler's Formula for Negative Powers

Although we have proved many criteria on the convergence of infinite series of numbers, seldom did we evaluate their sums. In this section we discuss a well-known summation formula for negative powers discovered by Euler in 1735, when he was twenty-eight.

From high school we learnt how to sum up a geometric progression. It led to the formula

$$\frac{1}{1-a} = 1+a+a^2+a^3+\cdots, \qquad a \in (-1,1).$$

In particular, taking $a=1/2$ and $-1/2$ yields

$$2 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots,$$

and

$$\frac{2}{3} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots .$$

From the previous sections we know more, for instance,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots ,$$

and

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Euler's formula gives a closed form for the sum

$$E_k = \sum_{n=1}^{\infty} \frac{1}{n^k},$$

when k is an even number. For instance, we have

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

Euler discovered this formula by a wonderful method based on analog thinking. To describe it we start with some simple facts on algebraic equations. Let a polynomial of degree n be given by

$$p(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Recall that x_0 is a root of p if $p(x_0) = 0$. The multiplicity of a root is defined to be the number m that satisfies $p(x_0) = p'(x_0) = \cdots = p^{(m-1)}(x_0) = 0$ but $p^{(m)}(x_0) \neq 0$. A root of multiplicity one is called a simple root and a root of multiplicity two is called a double root. The multiplicity appears in the power in the factorization of the polynomial. For instance, the polynomial $x^3 + x^2 - x - 1$ has a double root -1 and a simple root 1. We have the factorization

$$x^3 + x^2 - x - 1 = (x+1)^2(x-1).$$

Of course, a polynomial may admit complex roots so complete factorization over the real field is not always possible. For instance, for the polynomial $x^3 - 6x^2 + x - 6$ we stop at $(x^2+1)(x-6)$. However, suppose now that the polynomial of degree n $p(x)$ has exactly n many real simple roots $\alpha_1, \cdots, \alpha_n \neq 0$. By comparing the coefficients of the constant term we have the factorization formula

$$1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \prod_{k=1}^{n} \Big(1 - \frac{x}{\alpha_k}\Big).$$

By comparing the coefficients of $x^k, k = 1, \cdots, n$ from both sides, we get

$$\begin{aligned} a_1 &= -\sum_j \frac{1}{\alpha_j}, \\ a_2 &= \sum_{i<j} \frac{1}{\alpha_i \alpha_j}, \end{aligned}$$

and

$$a_k = (-1)^k \sum_{i_1 < \cdots < i_k} \frac{1}{\alpha_{i_1} \cdots \alpha_{i_k}}, \quad k = 1, \cdots, n,$$

in general.

Now, consider the Taylor expansion for the sine function,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

which is valid for all x in $\mathbb{R}$. The function

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

(set $\sin x/x = 1$ at $x = 0$) is smooth on $\mathbb{R}$. Euler boldly regarded $\sin x/x$ as a polynomial of infinite degree and asserted that all roots of $\sin x/x = 0$ are real, simple and given by $\pm k\pi, k \geq 1$. Moreover, parallel to the factorization above, one has

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x}{k\pi}\right)\left(1 + \frac{x}{k\pi}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right) \tag{0.2}$$

By comparing the coefficients of x^2 in this infinite product with the Taylor's series of $\sin x/x$, he got

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

By comparing the coefficients of x^4, he got

$$\frac{\pi^4}{90} = \sum_{k=1}^{\infty} \frac{1}{k^4}$$

after some manipulations. Going up step by step, all E_{2k} could be computed by looking at the coefficients of x^{2m} together with $E_{2(k-1)}, \cdots$, and E_2.

The following paragraphs are for optional reading.

The formula (0.2) was first obtained in a formal way. Years later, Euler justified it in rigorous terms. His proof used complex variables but the essential idea could be carried entirely out in the real field. There are other proofs using, for instance, Fourier series. The following "real" proof is taken from O. Hijab, Introduction to Calculus and Classical Analysis, Springer-Verlag, 2007. In this formula the sine function is replaced by the hyperbolic sine to the same effect.

Proposition 0.10. *

$$\frac{\sinh \pi x}{\pi x} = \prod_{n=1}^{\infty}\Big(1+\frac{x^2}{n^2}\Big), \quad \forall x \in \mathbb{R}. \tag{0.3}$$

(Set $\sinh \pi x/\pi x = 1$ at $x = 0$.) Recall that $\sinh x = (e^x - e^{-x})/2$ is the hyperbolic sine function.

Proof. We start with an identity of factorization: For $a, b > 0$,

$$a^{2n} - b^{2n} = (a^2 - b^2)\prod_{k=1}^{n-1}\big(a^2 - 2ab\cos\frac{k\pi}{n} + b^2\big)$$

(Exercise). So

$$\begin{aligned}
&\frac{\Big(1+\frac{\pi x}{2n}\Big)^{2n} - \Big(1-\frac{\pi x}{2n}\Big)^{2n}}{2\pi x} = \frac{\Big(1+\frac{\pi x}{2n}\Big)^{2} - \Big(1-\frac{\pi x}{2n}\Big)^{2}}{2\pi x} \times \\
&\prod_{k=1}^{n-1}\Big[\Big(1+\frac{\pi x}{2n}\Big)^2 - 2\Big(1+\frac{\pi x}{2n}\Big)\Big(1-\frac{\pi x}{2n}\Big)\cos\frac{k\pi}{n} + \Big(1-\frac{\pi x}{2n}\Big)^2\Big] \\
= \;& \frac{1}{n}\prod_{k=1}^{n-1}\Big[2\Big(1+\frac{\pi^2x^2}{4n^2}\Big) - 2\Big(1-\frac{\pi^2x^2}{4n^2}\Big)\cos\frac{k\pi}{n}\Big] \\
= \;& \frac{1}{n}\prod_{k=1}^{n-1}\Big[2\Big(1+\frac{\pi^2x^2}{4n^2}\Big)\big(\sin^2 x + \cos^2 x\big) - 2\Big(1-\frac{\pi^2x^2}{4n^2}\Big)\Big(\cos^2\frac{k\pi}{2n} - \sin^2\frac{k\pi}{2n}\Big)\Big] \\
= \;& \frac{1}{n}\prod_{k=1}^{n-1}\Big(4\sin^2\frac{k\pi}{2n} + \frac{\pi^2x^2}{n^2}\cos^2\frac{k\pi}{2n}\Big)
\end{aligned}$$

Letting $x \to 0$, an application of L'Hospital's rule yields

$$1 = \frac{1}{n}\prod_{k=1}^{n-1} 4\sin^2\frac{k\pi}{2n}.$$

By termwise division,

$$\frac{\Big(1+\frac{\pi x}{2n}\Big)^{2n} - \Big(1-\frac{\pi x}{2n}\Big)^{2n}}{2\pi x} = \prod_{k=1}^{n-1}\Big[1 + \frac{x^2}{k^2}\varphi\big(\frac{k\pi}{2n}\big)\Big]$$

where $\varphi(t) = t^2\cot^2 t$. Using $\tan t \geq t$, we see that $\varphi(t) \leq 1$ on $(0, \pi/2)$. Therefore,

$$\begin{aligned}
\frac{\Big(1+\frac{\pi x}{2n}\Big)^{2n} - \Big(1-\frac{\pi x}{2n}\Big)^{2n}}{2\pi x} \;&\leq\; \prod_{k=1}^{n-1}\Big(1+\frac{x^2}{k^2}\Big) \\
&\leq\; \prod_{k=1}^{\infty}\Big(1+\frac{x^2}{k^2}\Big).
\end{aligned}$$

Let $n \to \infty$,

$$\frac{\sinh \pi x}{\pi x} \le \prod_{k=1}^{\infty} \Big(1 + \frac{x^2}{k^2}\Big).$$

On the other hand, fix n_1 so that

$$\frac{\Big(1 + \frac{\pi x}{2n}\Big)^{2n} - \Big(1 - \frac{\pi x}{2n}\Big)^{2n}}{2\pi x} \ge \prod_{k=1}^{n_1 - 1} \Big(1 + \frac{x^2}{k^2}\varphi\big(\frac{k\pi}{2n}\big)\Big), \quad \forall n \ge n_1.$$

Letting $n \to \infty$,

$$\frac{\sinh \pi x}{\pi x} \ge \prod_{k=1}^{n_1 - 1} \Big(1 + \frac{x^2}{k^2}\Big)$$

and

$$\frac{\sinh \pi x}{\pi x} \ge \prod_{k=1}^{\infty} \Big(1 + \frac{x^2}{k^2}\Big)$$

follows by letting $n_1 \to \infty$.

$\square$

Taking log of both sides of (0.3) and using the continuity of the logarithmic function, we have

$$\begin{aligned}
\log \sinh \pi x - \log(\pi x) &= \log \lim_{n\to\infty} \prod_{k=1}^{n} \Big(1 + \frac{x^2}{k^2}\Big) \\
&= \lim_{n\to\infty} \log \prod_{k=1}^{n} \Big(1 + \frac{x^2}{k^2}\Big) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \log \Big(1 + \frac{x^2}{k^2}\Big) \\
&= \sum_{k=1}^{\infty} \log \Big(1 + \frac{x^2}{k^2}\Big).
\end{aligned}$$

We claim the series on the right hand side is uniformly convergent on $(0, M]$ for all $M > 0$. Indeed, by the mean-value theorem

$$\log \Big(1 + \frac{x^2}{k^2}\Big) = \frac{k^2}{k^2 + c^2}\frac{x^2}{k^2},$$

for some c between 1 and x^2/k^2. Therefore,

$$0 < \log \Big(1 + \frac{x^2}{k^2}\Big) \le \frac{M^2}{k^2}, \quad \forall x \in (0, M].$$

Taking $b_k = M^2/k^2$ and applying M-Test we obtain the result as claimed. Furthermore, the series obtained by differentiating $\sum_k \log(1 + x^2/k^2)$ whose general term is

$$\frac{k^2}{k^2+x^2}\frac{2x}{k^2}$$

also converges uniformly on every $(0, M]$. By the "exchange theorem" it is legal to differentiate both sides of

$$\log\sinh\pi x - \log\pi x = \sum_{k=1}^{\infty}\log\left(1+\frac{x^2}{k^2}\right)$$

to get

$$\pi\coth\pi x - \frac{1}{x} = \sum_{n=1}^{\infty}\frac{2x}{n^2+x^2}, \quad x \neq 0,$$

or

$$\frac{\pi\coth\pi\sqrt{x}}{\sqrt{x}} - \frac{1}{x} = \sum_{n=1}^{\infty}\frac{2}{n^2+x}, \quad x > 0,$$

We would like to expand the left hand side of this expression into a Taylor series. First, consider the function

$$\tau(x) = \begin{cases} \dfrac{x}{1-e^{-x}}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

This function is the reciprocal of the power series

$$\frac{1-e^{-x}}{x} = \sum_{k=0}^{\infty}(-1)^k\frac{x^k}{(k+1)!}, \quad \forall x \in \mathbb{R}.$$

We can expand it as a power series at 0,

$$\tau(x) = \sum_{n=0}^{\infty}\frac{\beta_n}{n!}\ x^n,$$

where $\beta_0 = 1,\ \beta_1 = 1/2,\ \beta_2 = 1/6,\ \beta_3 = 0,\ \beta_4 = -1/30, \cdots$, etc. Observing that

$$\frac{x}{2}\coth\frac{x}{2} = \tau(x) - \frac{x}{2},$$

we have

$$\frac{x}{2}\coth\frac{x}{2} = 1 + \sum_{n=2}^{\infty}\frac{\beta_n}{n!}\ x^n.$$

As the left hand side of this identity is an even function, $\beta_{2n+1} = 0, \forall n \geq 1$ and

$$\frac{x}{2}\coth\frac{x}{2} - 1 = \sum_{n=1}^{\infty}\frac{\beta_{2n}}{(2n)!}\ x^{2n}.$$

Finally we conclude

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\beta_{2n}}{(2n)!}(2\pi)^{2n}x^{n-1} = \sum_{n=1}^{\infty}\frac{1}{n^2+x}.$$

By differentiating both sides of this identity $k-1$ many times and then setting $x=0$ (uniform convergence after differentiation is easy to verify), we finally obtain

Theorem 0.11 (Euler's Formula). ** For all $k \geq 1$,*

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{(-1)^{k-1}}{2}\frac{\beta_{2k}}{(2k)!}(2\pi)^{2k}.$$

The number β_k is called the k-th Bernoulli number for $k \geq 2$. Despite the effort of many mathematicians, little is known for E_k when k is odd. It was proved as late as 1979 that E_3 is an irrational number. You may look up the expository paper, "Euler and his work on infinite series", Bulletin of AMS, 515-539, 2007, by V.S. Varadarajan for more.

It is interesting to observe that E_k are special values of the Riemann zeta function ζ defined by

$$\zeta(x) = \sum_{n=1}^{\infty}\frac{1}{n^x}, \quad x > 1.$$

It is not hard to see that this series converges uniformly on $[a,\infty)$ for each $a>1$ and ζ is smooth on $(1,\infty)$. (In fact, the zeta function can be defined in $z \in \mathbb{C}/\{1\}$.) Thus we have $E_k = \zeta(k)$. It is related to the Gamma function Γ by the relation

$$\zeta(x) = \frac{1}{\Gamma(x)}\int_0^{\infty}\frac{t^{x-1}}{e^t-1}dt.$$

Using this relation one could obtain another proof of Euler's formula. Finally, we point out that the the zeta function has deep relationship with prime numbers. The following identity was found by Euler:

$$\zeta(x) = \frac{1}{\prod_p\left(1-\dfrac{1}{p^x}\right)},$$

where the product is taken over all prime numbers. As $\lim_{x\to 1^+}\zeta(x)=\infty$, this identity shows that there are infinitely many prime numbers. This result essentially opens up a new branch of mathematics called analytic number theory.